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Weak Riemannian Structures on Gauge-Group Orbits

Vittorio Berzi[1] **and Marco Reni**[1]

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It has been shown recently that the orbit space for the (Sobolev extended) gauge group action admits a stratification into Hilbert manifolds. Here it is shown that these manifolds carry a natural weak Riemannian structure defined by a metric that corresponds to the kinetic part of the Lagrangian considered in heuristic Yang–Mills theories.

1. INTRODUCTION

Consider a pure Yang–Mills theory Y modeled in terms of a smooth principal bundle P. The space of Yang–Mills potentials of Y is the affine space $\mathscr{C}$ of connection forms of P and the gauge group is the group $\mathscr{G}$ of the automorphisms of P leaving fixed the points of the base manifold. There is a natural right action $\mathscr{M}$ of $\mathscr{G}$ on $\mathscr{C}$ induced by pullback.

Assume that the base of P is a (finite-dimensional) compact, connected, Riemannian, oriented manifold and that the structure group of P is a compact, real Lie subgroup of $GL(m, \mathbb{C})$, $m > 1$. Then for a sufficiently large integer k a Hilbert manifold $\mathscr{C}^k$ and a Hilbert Lie group $\mathscr{G}^{k+1}$ are defined, containing, respectively, $\mathscr{C}$ and $\mathscr{G}$ as dense subsets.

By the way they are defined $\mathscr{C}^k$ and $\mathscr{G}^{k+1}$ are said to be the Sobolov extended space of connections of order k and the Sobolev extended gauge group of order $k+1$.

The action $\mathscr{M}$ of $\mathscr{G}$ on $\mathscr{C}$ can be extended by continuity to a smooth action $\mathscr{M}$ of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$.

It is reasonable to interpret the elements of $\mathscr{C}^k$ as defining the field configurations of Y, regarded as a classical field theory. Then the gauge invariance of the theory implies that any two elements of $\mathscr{C}^k$ lying in the same orbit for the action of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$ represent the *same* configuration. Thus, one is led to conjecture that the quotient space $\mathscr{C}^k/\mathscr{G}^{k+1}$ or some

[1]Istituto di Scienze Fisiche dell'Università, Milan, Italy.

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subset of it can be taken as the "true" configuration space of Y, in view of a possible Lagrangian-Hamiltonian formulation of Y (see Daniel and Viallet, 1980; Babelon and Viallet, 1981). For this it is necessary that $\mathscr{C}^k/\mathscr{G}^{k+1}$ or a suitable subset of it can be given a smooth quotient manifold structure and even a (possibly weak) Riemannian structure. This is possible if $\mathscr{C}^k$ is replaced by a certain $\mathscr{G}^{k+1}$-invariant subset $\mathscr{C}_0^k \subset \mathscr{C}^k$ (the space of generic connections) and $\mathscr{G}^{k+1}$ by the quotient group $\mathscr{G}^{k+1}/Z$, where Z is the center of $\mathscr{G}^{k+1}$, which is a finite group if one assumes that G is semisimple [see Narasimhan and Ramadas (1979) for the case of a trivial principal bundle and Mitter and Viallet (1981) for the general case; compare also Singer (1978) and Atiyah et al. (1978)]. A more general result has been given in Kondracki and Rogulski (1983) by showing that the topological space $\mathscr{C}^k/\mathscr{G}^{k+1}$ admits a stratification $\mathscr{S}$ into Hilbert manifolds. In more detail, one has a partition of $\mathscr{C}^k$ into a countable family $(\mathscr{C}_\alpha^k)$ of $\mathscr{G}^{k+1}$-invariant submanifolds and for each α the orbit manifold $\mathscr{C}_\alpha^k/\mathscr{G}^{k+1}$ exists. Moreover, for a convenient α_0, $\mathscr{C}_{\alpha_0}^k = \mathscr{C}_0^k$ is an open, dense subset of $\mathscr{C}^k$.

The natural question at this stage is whether one can give the quotient manifolds $\mathscr{C}_\alpha^k/\mathscr{G}^{k+1}$ a (possibly weak) Riemannian structure. This structure should be natural from the mathematical viewpoint and it should be physically meaningful when we regard $\mathscr{C}_0^k/\mathscr{G}^{k+1}$ as the configuration manifold of a Yang-Mills theory.

In this paper we give a positive answer to the question by introducing on the $\mathscr{C}_\alpha^k/\mathscr{G}^{k+1}$ and in particular on $\mathscr{C}_0^k/\mathscr{G}^{k+1}$ a weak metric, which corresponds to the "kinematic" part of the Lagrangian considered in the heuristic formulation of Yang-Mills theories. The content of this paper is as follows. We first review in some detail the definitions and the results sketched above [in a formulation substantially equivalent to the one in Kondracki and Rogulski (1983)]. This will give us the appropriate setting for the construction of weak matrices on the orbit manifold. The paper is concluded by a preliminary study of the geodesic structure of the generic orbit manifold.

2. THE GAUGE GROUP AND ITS ACTION ON CONNECTION FORMS

The kinematics of a Yang-Mills theory can be formulated by assuming as basic object a smooth principal bundle $P = (R, p, M, G)$ with total space R, projection p, base manifold M, and structure group G.

In this paper we shall assume that M is a finite-dimensional, connected, compact, Riemannian, oriented manifold and G is a compact, real Lie subgroup of $GL(m, \mathbb{C})$ for some $m > 1$. The Lie algebra g of G is always identified with a real Lie subalgebra of $M(m, \mathbb{C})$.

The fundamental concepts of the theory are those of the space of connections (physically the space of Yang–Mills potentials) and of the gauge group. There are several (algebraically) equivalent mathematical objects describing these notions. The most directly related to P are: (a) for the gauge group, the group $\mathscr{G}_2$ of the M-automorphisms of P; and (b) for the space of connections, the set $\mathscr{C}_2$ of principal connection forms of P.

$\mathscr{C}_2$ is in fact an affine space contained in the vector space of g-valued smooth 1-forms on R.

Physically the gauge group is introduced in the form of a group of transformations of the Yang–Mills potentials under which the theory is invariant.

Correspondingly, in the mathematical model, we have a right action

$$\mathscr{M}_2: \quad \mathscr{C}_2 \times \mathscr{G}_2 \to \mathscr{C}_2: \quad (\omega, f) \rightsquigarrow \omega \cdot f = f^*(\omega)$$

of $\mathscr{G}_2$ on $\mathscr{C}_2$, induced by pullback.

We now define two sequences of objects isomorphic respectively to $\mathscr{G}_2$ and to $\mathscr{C}_2$. Consider first the subset $\mathscr{G}_2'$ of $C^\infty(R, G)$ consisting of the maps $h: R \to G$ satisfying

$$h(r \cdot a) = a^{-1} h(r) a, \qquad r \in R, \quad a \in G$$

$\mathscr{G}_2'$ is a group by pointwise multiplication and there is a unique group (anti)isomorphism $\beta_2': \mathscr{G}_2 \to \mathscr{G}_2'$ such that

$$r \cdot \beta_2'(f)(r) = f(r), \qquad r \in R$$

The next step in our sequence requires the following notations. Let $\mathscr{U}$ be the set of the open subsets $U \subset M$, $U \neq \varnothing$, such that P and M are trivializable on U. For each $U \in \mathscr{U}$ let Σ_U be the set of smooth sections of P over U and let $\Sigma = \bigcup_{U \in \mathscr{U}} \Sigma_U$. We shall denote by dom σ the definition set of a $\sigma \in \Sigma$. Consider the subset $\mathscr{G}_1$ of $\Pi_{\sigma \in \Sigma} C^\infty(\text{dom}\,\sigma, G)$ consisting of the families $(\zeta_0)_{\sigma \in \Sigma}$ such that if $\sigma, \sigma' \in \Sigma_U$, then

$$\text{(i)} \quad \zeta_\sigma \,|\, U = \zeta_{\sigma|U}$$

$$\text{(ii)} \quad \zeta_{\sigma'} = \lambda^{-1} \zeta_\sigma \lambda$$

where λ is the unique smooth map $U \to G$ such that $\sigma' = \sigma\lambda$. The group $\mathscr{G}_2$ is a group for the composition $(\zeta_\sigma)(\zeta_{\sigma'}) = (\zeta_\sigma \zeta_{\sigma'})$ and there is a unique isomorphism $\beta_2: \mathscr{G}_2' \to \mathscr{G}_1$ such that

$$\beta_2(h)_\sigma = h \circ \sigma, \qquad \sigma \in \Sigma$$

Similarly, consider the subset $\mathscr{C}_1$ of the vector space $\Pi_\sigma A^1(\text{dom}\,\sigma, g)$ consisting of families $(\mathscr{A}_\sigma)$ satisfying

$$\text{(i)} \quad \mathscr{A}_\sigma \,|\, U = \mathscr{A}_{\sigma|U}$$

$$\text{(ii)} \quad \mathscr{A}_{\sigma'} = \lambda^{-1} \mathscr{A}_\sigma \lambda + \lambda^{-1} T\lambda$$

for σ, $\sigma' \in \Sigma_U$ and $\sigma' = \sigma \cdot \lambda$ [here we are adopting the notation $A^k(U, F)$ for the vector space of k-forms over an open set U of a manifold with values in a finite-dimensional real vector space F].

It is immediate that $\mathscr{C}_2$ is an affine subspace of $\Pi_\sigma A^1(\operatorname{dom} \sigma, g)$ and it is well known that the map

$$\alpha_2\colon \quad \mathscr{C}_2 \to \mathscr{C}_1, \qquad \alpha_2(\omega)_\sigma = \sigma^*(\omega), \quad \sigma \in \Sigma$$

is a bijection and in fact an isomorphism of affine spaces.

To proceed to the next step, we introduce the associated bundle $R \times {}^G M(m, \mathbb{C})$ defined by the right operation $(a, s) \rightsquigarrow s^{-1}as$ of G in $M(m, \mathbb{C})$. We shall pose $F = M(m, \mathbb{C})$ and we shall denote by E_F the above-defined associated bundle. F will be always regarded as a real associative algebra.

It is not difficult to see that E_F is an associative algebra bundle. In fact, denoting by $r \cdot a$ the element of E_F corresponding to $(r, a) \in R \times F$ and by p_F the projection of E_F, we have a natural associative algebra structure on each fiber $p_F^{-1}(x)$, $x \in M$, such that

$$\lambda(r \cdot a) = r \cdot (\lambda a)$$

$$r \cdot a + r \cdot a' = r \cdot (a + a')$$

$$(r \cdot a)(r \cdot a') = r \cdot aa'$$

$$r \in p^{-1}(x), \qquad \lambda \in \mathbb{R}, \qquad a, a' \in F$$

Moreover, given a section $\sigma \in \Sigma_U$, with $x \in U$, the map

$$\{x\} \times F \to p_F^{-1}(x), \qquad (x, a) \rightsquigarrow \sigma(x) \cdot a$$

is a diffeomorphism and an associative algebra isomorphism. If we choose for every $x \in M$ an element $r_x \in p^{-1}(x)$, then the sets $r_x \cdot G$ (resp. $r_x \cdot g$) are the fibers of a subbundle E_G (resp. E_g) of E_F. Then E_G (resp. E_g) is isomorphic to the associated bundle $R \times {}^G G$ (resp. $R \times {}^G g$) defined by the action of G in G (resp. of G in g) induced by the action of G in F. As for E_F, one sees that E_G (resp. E_g) is a Lie-group bundle (resp. a Lie-algebra bundle): each fiber $p_G^{-1}(x)$ of E_G [resp. $p_g^{-1}(x)$ of E_g] has a natural Lie-group (resp. Lie-algebra) structure depending smoothly on x. For every $x \in M$, $r \in p^{-1}(x)$, and $s, s' \in G$ (resp. $u, u' \in g$) the group product of $r \cdot s$ and $r \cdot s'$ (resp. the Lie product of $r \cdot u$ and $r \cdot u'$) is given by $r \cdot ss'$ {resp. $r \cdot [u, u'] = r \cdot (uu' - u'u)$}. Further, $p_g^{-1}(x)$ can be identified with the Lie algebra of $p_G^{-1}(x)$.

Now denote $C^\infty(\xi, V)$ the set of smooth sections of a smooth bundle ξ over V and put $C^\infty(\xi) = C^\infty(\xi, B)$ if B is the base manifold of ξ. We remark that in view of preceding considerations $C^\infty(E_G)$ has a natural group structure by pointwise multiplication. Then define $\mathscr{G}$ to be the group

$C^\infty(E_G) \subset C^\infty(E_F)$. By standard properties of associated bundles one checks easily that there is a group isomorphism $\beta_1 : \mathscr{G}_1 \to \mathscr{G}$ such that

$$\beta_1(\zeta)(x) = \sigma(x) \cdot \zeta_\sigma(x), \qquad x \in \operatorname{dom} \sigma, \qquad \sigma \in \Sigma$$

Finally, consider the subset $\mathscr{C}$ of $\Pi_{\sigma\in\Sigma} L(T(\operatorname{dom}\sigma), E_g)$ consisting of the families (A_σ) such that

$$A_\sigma(x)(h_x) = \sigma(x) \cdot (\mathscr{A}_\sigma(x)(h_x)), \qquad x \in \operatorname{dom}\sigma, \quad \sigma \in \Sigma, \quad h_x \in T_xM$$

for some $\mathscr{A} = (\mathscr{A}_\sigma) \in \mathscr{C}_1$. The subset $\mathscr{C}$ is an affine subspace of the vector space $\Pi_\sigma L(T(\operatorname{dom}\sigma), E_g)$ and we have an affine isomorphism

$$\alpha_1: \quad \mathscr{C}_1 \to \mathscr{C}$$

$$\alpha_1(\mathscr{A})_\sigma(x)(h_x) = \sigma(x) \cdot \mathscr{A}_\sigma(h_x)$$

In conclusion, we have a canonical group anti-isomorphism

$$\beta = \beta_1\beta_2'\beta_2: \quad \mathscr{G}_2 \to \mathscr{G}$$

and a canonical affine space isomorphism

$$\alpha = \alpha_2\alpha_1: \quad \mathscr{C}_2 \to \mathscr{C}$$

The right action of $\mathscr{G}_2$ in $\mathscr{C}_2$ can be shifted to a right action of $\mathscr{G}$ in $\mathscr{C}$,

$$\mathscr{M}: \quad \mathscr{C} \times \mathscr{G} \to \mathscr{C}$$

by requiring that the diagram

$$\begin{array}{ccc} \mathscr{C}_2 \times \mathscr{G}_2 & \xrightarrow{\mathscr{M}_2} & \mathscr{C}_2 \\ {\scriptstyle \alpha\times\beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathscr{C} \times \mathscr{G} & \xrightarrow{\mathscr{M}} & \mathscr{C} \end{array}$$

commutes. With some computations one finds

$$\mathscr{M}(A, g)_\sigma = (A \cdot g)_\sigma = A_\sigma + g_\sigma^{-1}\nabla_A g_\sigma, \qquad \sigma \in \Sigma, \quad A \in \mathscr{C}, \quad g \in \mathscr{G} \quad (1)$$

where $g_0 = g\,|\operatorname{dom}\sigma$ and $\nabla_A : C^\infty(E_F) \to C^\infty(L(TM, E_F))$ is the exterior covariant differential associated with the connection form $\alpha^{-1}(A) \in \mathscr{C}_2$.

3. ISOTROPY GROUPS OF THE GAUGE ACTION

The action of the gauge group on connection froms is not at all free. Consider first the action $\mathscr{M}_2$: $\mathscr{C}_2 \times \mathscr{G}_2 \to \mathscr{C}_2$. The isotropy subgroups $I_\omega \subset \mathscr{G}_2$, $\omega \in \mathscr{C}_2$, are closely related to the holonomy groups of ω [for the definition and theory of holonomy groups of a principal connection, see Kobayashi and Nomizu (1963)]. Let ω be an element of $\mathscr{C}_2$ and denote by $\phi(r)$, $r \in R$, the holonomy group of ω at r [$\phi(r)$ is a Lie subgroup of G and, since M is assumed to be connected, the holonomy groups $\phi(r)$ $r \in R$, are all conjugated to each other in G]. Then we have the following standard result:

Theorem 3.1. Let $f \in \mathscr{G}_2$ and write $f(r) = r \cdot g(r)$, $r \in R$. Then $f^*\omega = \omega$, i.e., f is in the isotropy subgroup of $\mathscr{G}_2$ at ω for the action $\mathscr{M}_2$, if and only if the following conditions hold:

(i) For some $r_0 \in R$, $g(r_0)$ belongs to the centralizer of $\phi(r_0)$ in G.
(ii) For every piecewise smooth path $\lambda : [0, 1] \to M$ with $\lambda(0) = p(r_0)$, one has $g(\lambda_{r_0}(1)) = g(r_0)$, where λ_{r_0} is the horizontal lift of λ at r_0.

Moreover, if (i) and (ii) are satisfied, then $g(r)$ is in the centralizer of $\phi(r)$ $\forall r \in R$.

A similar result can be obtained for the action $\mathscr{M} : \mathscr{C} \times \mathscr{G} \to \mathscr{C}$. Let A be a fixed element of $\mathscr{C}$ and let I_A be the isotropy subgroup of $\mathscr{G}$ for $\mathscr{M}$ at A. It is clear from equation (1) that $g \in I_A$ iff $\nabla_A g = 0$. This equation is equivalent to conditions similar to (i) and (ii) of Theorem 3.1. To see this, notice that for $r \in R$ we can define a map $\chi_r : \mathscr{G} \to G$ by requiring that

$$r \cdot \chi_r(g) = g(p(r)), \qquad \forall g \in \mathscr{G}$$

In fact, χ_r is a group homomorphism. Now let $\phi(r)$, $r \in R$, be the holonomy group at r of the principal connection on P defined by $\alpha^{-1}(A)$; then we have the following analog of Theorem 3.1:

Theorem 3.2. $g \in I_A$ if and only if the following conditions are satisfied:

(i) For some $r_0 \in R$, $\chi_{r_0}(g)$ belongs to the centralizer of $\phi(r_0)$ in G.
(ii) For every piecewise smooth path $\lambda : [0, 1] \to M$ with $\lambda(0) = p(r_0)$ one has

$$\chi_{\lambda_{r_0}}(1)(g) = \chi_{r_0}(g)$$

where λ_{r_0} is the horizontal lift (in R) of λ.

If (i) and (ii) are satisfied, then for all $r \in R$, $\chi_r(g)$ belongs to the centralizer of $\phi(r)$ in G.

Condition (i) of Theorem 3.2 tells us that if $g \in I_A$, then g is determined by its value at any point $x \in M$. We have in fact a more refined result, which can be obtained by using the Reduction Theorem of Kobayashi and Nomizu (1963):

Theorem 3.3. With the notations above for any $r \in R$, the map $\chi_r : \mathscr{G} \to G$ induces a group isomorphism of I_A onto the centralizer $\mathscr{Z}(\phi(r))$ of $\phi(r)$ in G.

4. SOBOLEV EXTENSIONS OF THE GAUGE GROUP, OF THE SPACE OF CONNECTION FORMS, AND OF THE GAUGE GROUP ACTION

If we want to construct a "true" configuration space for a Yang-Mills theory as a quotient of connections space by the gauge group action in a differential geometric setting, then the problem arises of giving $\mathcal{G}$ and $\mathcal{C}$ smooth manifold structures in such a way that $\mathcal{G}$ becomes a Lie group and the action of $\mathcal{G}$ on $\mathcal{C}$ turns out to be smooth.

This is possible if we admit for $\mathcal{G}$ and $\mathcal{C}$, manifold structures modeled on suitable classes of locally convex vector spaces: see Cirelli and Manià (1985) and Abbati et al. (1987) for a realization of such a program.

However, from the viewpoint of a geometrization of the Yang-Mills theories, we need to have on the "true" configuration space something more than a smooth structure, that is, a (possibly weak) Riemannian structure. Thus, it seems unavoidable to introduce the so-called Sobolev extensions of the gauge group $\mathcal{G}$ and of the connections space $\mathcal{C}$. In this way one gets smooth manifolds modeled on (separable) Hilbert spaces and the possibility of introducing (weak) Riemannian structures is left open.

We sketch here the main points concerning the construction of Sobolev extensions of the three basic ingredients of the theory, namely $\mathcal{G}$, $\mathcal{C}$, and the gauge group action $\mathcal{M}$. We adopt the intrinsic formulations of Palais (1965, 1968).

4.1. Extension of the Gauge Group

A real Hilbert space structure on $F = M(m, \mathbb{C})$ is defined by the scalar product

$$(a \mid b)_F = \int \operatorname{Re} \operatorname{Tr}[tat^{-1}(tbt^{-1})^*] \, dh(t)$$

where dh is the normalized right-invariant Haar measure on G. Notice that for $G = SU(n)$ one has simply

$$(a \mid b)_F = \operatorname{Tr} ab^*$$

The scalar product induces a Riemannian structure $x \rightsquigarrow (\,|\,)_x$ on E_F given by

$$(r \cdot a \mid r \cdot b)_x = (a \mid b)_F, \qquad a, b \in F, \quad x \in M, \quad r \in p^{-1}(x)$$

By Palais (1965, Chapter IX) we can associate to the Riemannian vector bundle E_F a sequence (discrete Sobolev chain) $H^s(E_F)$, $s = 0, 1, \ldots,$ of Hilbertable vector spaces which look locally like Sobolev spaces of type L^2_s.

Here the term "Hilbertable" space means a topological vector space whose topology can be defined by a Hilbert scalar product. But for $s > 0$

this Hilbert scalar product cannot be chosen in a canonical way. Instead, on $H^0(E_F)$ there is a canonical compatible scalar product $(|)_0$ induced by the Riemannian structure of E_F and given by

$$(f|g)_0 = \int (f(x)|g(x))_x \, d\mu_v, \qquad f, g \in H^0(E_F)$$

where μ_v is the Lebesgue measure defined by the volume form of M. The integral converges, since $H^0(E_F)$ consists of the measurable sections f of E_F such that

$$\int (f(x)|f(x))_x \, d\mu_v < \infty$$

Each $H^s(E_F)$ can be identified with a subset of the set of measurable sections of E_F; with this identification, $H^s(E_F) \supset H^{s+1}(E_F)$, $s = 0, 1, \ldots$. Moreover, $H^s(E_F)$ is a dense subset of $H^t(E_F)$ for $s \geq t$, and the topology of $H^s(E_F)$ is finer than the topology induced from $H^t(E_F)$. Finally, and this is essential for the subsequent developments, $C^\infty(E_F)$ is a dense subset of $H^s(E_F)$ for every $s = 0, 1, 2, \ldots$. On each space $H^s(E_F)$ there is always a Hilbert scalar product $(|)_s$ compatible with its topological vector space structure.

Since E_F is an associative algebra-bundle, we have an induced associative algebra structure of the set $\Gamma(E_F)$ of all measurable cross section of E_F, the algebra operations being defined pointwise. The spaces $H^s(E_F)$ are vector subspaces of $\Gamma(E_F)$, but in general are not subalgebras, i.e., they are not closed for pointwise multiplication. Still, one has the following result, which can be obtained by a straightforward generalization of Palais (1968), Corollary 9.7:

Theorem 4.1. For $s > \frac{1}{2} \dim M$, $H^s(E_F)$ is a Hilbertable algebra for pointwise multiplication and $\forall 0 \leq J \leq s$, $H^J(E_F)$ is a topological $H^s(E_F)$-module.

By definition, $\mathscr{G} = C^\infty(E_G) \subset C^\infty(E_F) \subset H^s(E_F)$ for $s = 0, 1, 2, \ldots$. Hence, for every s we can consider the closure $\mathscr{G}^s$ of $\mathscr{G}$ in $H^s(E_F)$. Now let k be the least integer such that $k > \frac{1}{2} \dim M + 1$. We shall say that $\mathscr{G}^{k+1}$ is the Sobolev extension of the gauge group $\mathscr{G}$.

Since G is compact, it is a closed submanifold of the real manifold $F = M(m, \mathbb{C})$. Hence, combining the "Mayer–Vietoris theorem" of Palais (1968, Section 4) with the result of Eells (1966, p. 781), one gets:

Theorem 4.2. G^{k+1} is a smooth, closed submanifold of $H^{k+1}(E_F)$.

Since in a Banach algebra B the set B^* of invertible elements is an open submanifold and $f \rightsquigarrow f^{-1}$ is a diffeomorphism $B^* \to B^*$, from Theorems 4.1 and 4.2 we have the following result:

Corollary 4.3. The extended gauge group $\mathscr{G}^{k+1}$ is a Hilbertable Lie group for the manifold structure induced from $H^{k+1}(E_F)$.

Remark. By Theorem 4.2 and the preceding corollary there is a well defined smooth map

$$\mathscr{G}^{k+1}\times H^J(E_F)\to H^J(E_F):\quad (g,f)\rightsquigarrow gfg^{-1},\qquad 0\le J\le k$$

One checks that the canonical scalar prooduct $(\,|\,)_0$ is invariant under this map. Further, for $0\le J\le k$ one can choose a scalar product $(\,|\,)_J$ on $H^J(E_F)$ having the same invariance property. Briefly, we shall say that the scalar products $(\,|\,)_J$ are $\mathscr{G}^{k+1}$-invariant.

Taking account of the definition of E_g, it is not difficult to see that the Lie algebra of $\mathscr{G}^{k+1}$ can be identified with $H^{k+1}(E_g)$ [by the choice of k and Theorem 4.1, $H^{k+1}(E_g)$ has a natural Lie algebra structure, the Lie product being defined by $[u,v]=uv-vu$].

Moreover, the exponential map of Lie$(\mathscr{G}^{k+1})$ into G^{k+1} can be identified with the map

$$H^{k+1}(E_g)\to G^{k+1},\qquad u\rightsquigarrow e^u=\sum_{n=0}^{\infty}\frac{u^n}{n\,!}$$

4.2. Extension of the Space of Connection Forms

Let $\mathscr{D}_1$ be the set of differences $A'-A''$, $A',A''\in\mathscr{C}$. From the definition of $\mathscr{C}$ in Section 2 it follows that $\mathscr{D}_1$ is a vector space. Thus, $\mathscr{C}$ is an affine space over $\mathscr{D}_1$ and for $A\in\mathscr{C}$ we have a bijection ("affine coordinate map"):

$$\lambda_A:\quad \mathscr{C}\to\mathscr{D}_1\ ,\quad A'\to A'-A$$

If $A',A''\in\mathscr{C}$ and $A'=(A'_\sigma)$, $A''=(A''_\sigma)$, then $A'-A''=(A'_\sigma-A''_\sigma)$. Again from the definition of $\mathscr{C}$ it follows that $(A_\sigma)\in\Pi L(T(\text{dom}\,\sigma),E_g)$ is in $\mathscr{C}$ if and only if the following condition is satisfied

(c) Let $U\in\mathscr{U}$ and $\sigma,\sigma'\in\Sigma_U$; then
1. $A_\sigma\,|\,U=A_{\sigma|U}$.
2. If $\lambda:U\to G$ is defined by $\sigma'=\sigma\lambda$, then

$$A_{\sigma'}=(\sigma\cdot\lambda)^{-1}A_\sigma(\sigma\cdot\lambda)+\sigma\cdot(\lambda^{-1}T\lambda)\qquad (*)$$

One concludes that $\mathscr{D}_1\subset\Pi\ L(T\,\text{dom}\,\sigma,E_g)$ and $(\tilde{B}_\sigma)\in\Pi L(T(\text{dom}\,\sigma),E_g)$ is in $\mathscr{D}_1$ if and only if it satisfies the condition obtained from (c) by replacing equation $(*)$ with the equation

$$\tilde{B}_{\sigma'}=(\sigma\cdot\lambda)^{-1}\tilde{B}_\sigma(\sigma\cdot\lambda)\qquad (**)$$

This imples that there is a linear isomorphism

$$\gamma:\quad \mathscr{D}_1\to\mathscr{D}=C^\infty(L(TM,E_g))$$

such that

$$\gamma(\tilde{B})\,|\,\mathrm{dom}\,\sigma = \tilde{B}_\sigma, \qquad \sigma \in \Sigma$$

Hence for each $A \in \mathscr{C}$ we have a bijective map (affine coordinate map)

$$\varphi'_A = \gamma \circ \lambda_A\colon \quad \mathscr{C} \to \mathscr{D}$$

Now we observe that there is a canonical isomorphism of vector bundles:

$$L(TM, E_F) \approx E_F \otimes T^*M$$

This induces a linear isomorphism:

$$C^\infty(L(TM, E_F)) \approx C^\infty(E_F \otimes T^*M)$$

We shall regard the preceding isomorphisms as identifications. This is useful since $E_F \otimes T^*M$ has a canonical Riemannian bundle structure, as follows.

Since M is an oriented Riemannian manifold, we have a canonical Riemannian bundle structure on T^*M: $x \rightsquigarrow (\,|\,)_{M,x}$ defined by the equation

$$(\alpha_x\,|\,\beta_x)_{M,x} V(x) = \alpha_x \wedge {}^*\beta_x$$

for $\alpha_x, \beta_x \in T^*_x M$, where V is the volume form of M and $*$ is the Hodge operator. The Riemannian bundle structure $x \rightsquigarrow (\,|\,)_x$ on $E_F \otimes T^*M$ is given by

$$(r \cdot a \otimes \alpha_x\,|\,r \cdot b \otimes \beta_x)_x = (a\,|\,b)_F (\alpha_x\,|\,\beta_x)_{M,x}$$

for $\alpha_x, \beta_x \in T^*_x M$, $r \in p^{-1}(x)$, and $a, b \in F$, with $(a\,|\,b)_F$ being the scalar product on F defined in Section 4.1.

Then we have, as above, Sobolev chains of Hilbertable space

$$H^s(E_F \otimes T^*M), \qquad H^s(E_g \otimes T^*M), \qquad s = 0, 1, 2, \ldots$$

Again for $s > 0$ there is not a canonical scalar product, while on $H^0(E_F \otimes T^*M)$ and on $H^0(E_g \otimes T^*M)$ there is a canonical scalar product

$$(f\,|\,g)_0 = \int (f(x)\,|\,g(x))_x \, d\mu_v \tag{2}$$

Let $k > \frac{1}{2}\dim M + 1$ be the integer introduced in Section 4.1. Then, since E_g is a closed submanifold of E_F, by the result of Eells (1966) quoted above, $H^*(E_g \otimes T^*M)$ is a closed subspace of $H^k(E_F \otimes T^*M)$. We shall pose

$$\mathscr{D}^k = H^k(E_g \otimes T^*M)$$

Now let $\mathscr{O}$ be any element of $\mathscr{C}$. Put $\xi = L(T^*M, E_g) \simeq E_g \otimes T^*M$ and denote by $\Gamma(\xi\,|\,V)$ the set of measurable sections of ξ over V. Put

$$\mathscr{C}^k_{\mathscr{O}} = \left\{ A \in \prod_{\sigma \in \Sigma} \Gamma(\xi\,|\,\mathrm{dom}\,\sigma)\,\middle|\, A_\sigma = \mathscr{O}_\sigma + B\,|\,\mathrm{dom}\,\sigma, \quad B \in D^k \right\}$$

It is clear that $\mathscr{C}^k_{\mathcal{O}} \supset \mathscr{C}$ and it is easily seen that $\mathscr{C}^k_{\mathcal{O}}$ is independent of $\mathcal{O} \in \mathscr{C}$. Put

$$\mathscr{C}^k = \mathscr{C}^k_{\mathcal{O}} \qquad \text{and} \quad \mathcal{O} \in \mathscr{C}$$

We shall say that $\mathscr{C}^k$ is the extended space of connection forms. We have a bijection

$$\varphi_{\mathcal{O}}\colon \quad \mathscr{C}^k \to \mathscr{D}^k, \qquad \varphi_{\mathcal{O}}(A(|\mathrm{dom}\,\sigma = A_\sigma - \mathcal{O}_\sigma, \quad \sigma \in \Sigma$$

such that $\varphi_{\mathcal{O}} | \mathscr{C} = \varphi'_{\mathcal{O}}$, where $\varphi'_{\mathcal{O}}$ is the affine coordinate map $\gamma \circ \lambda_{\mathcal{O}} : \mathscr{C} \to \mathscr{D}$ introduced above.

The bijection $\varphi_{\mathcal{O}}$ is itself an affine coordinate map and there is a unique topological structure $\mathscr{T}_{\mathcal{O}}$ on $\mathscr{C}^k$, compatible with its affine space structure and such that $\varphi_{\mathcal{O}}$ is a homeomorphism. One sees easily that $\mathscr{T}_{\mathcal{O}} = \mathscr{T}$ is independent of $\mathcal{O}$ and $\mathscr{C}$ is dense in $\mathscr{C}^k$. Since the Hilbertable space $\mathscr{D}^k$ is trivially a Hilbertable smooth manifold, it is clear that $\mathscr{C}^k$ has a unique Hilbertable smooth manifold structure such that for every $\mathcal{O} \in \mathscr{C}$ the triple $c_{\mathcal{O}} = (\mathscr{C}^k, \varphi_{\mathcal{O}}, \mathscr{D}^k)$ is a chart for this structure. We shall say that the manifold $\mathscr{C}^k$ so defined is the manifold of Sobolev extended connection forms, or, simply, the manifold of connection forms.

In the next section we shall seee that the action of $\mathscr{G}$ on $\mathscr{C}$ extends uniquely to a smooth action of the extended gauge group on the manifold $\mathscr{C}^k$.

4.3. Extension of the Gauge Group Action

Let $\mathcal{O} \in \mathscr{C}$ be fixed. The action $\mathcal{M} : \mathscr{C} \times \mathscr{G} \to \mathscr{C}$ given by (1) can be shifted to an action $\mathcal{N}$: of $\mathscr{G}$ in $\mathscr{D}$ by requiring that the diagram

$$\begin{array}{ccc} \mathscr{C} \times \mathscr{G} & \xrightarrow{\ \mathcal{M}\ } & \mathscr{C} \\ \big\downarrow {\scriptstyle \varphi'_{\mathcal{O}} \times 1} & & \big\downarrow {\scriptstyle \varphi_{\mathcal{O}}} \\ \mathscr{D} \times \mathscr{G} & \xrightarrow{\ \mathcal{N}_{\mathcal{O}}\ } & \mathscr{D} \end{array}$$

is commutative. With some computations one gets

$$\mathcal{N}_{\mathcal{O}}(B, g) =: B \cdot g = g^{-1} B g + g^{-1} \nabla_{\mathcal{O}} g, \qquad B \in \mathscr{D}, \quad g \in \mathscr{G} \tag{3}$$

Equivalently one has

$$B \cdot g = B + g^{-1} \nabla_{\varphi_{\mathcal{O}}^{-1}(B)} g$$

Now, with the notations of Section 4.2, there is a bilinear morphism of vector bundles $E_F \oplus (E_F \otimes T^*M) \to E_F \otimes T^*M$ (where $\oplus$ denotes the Whitney sum) such that

$$(r \cdot a, r \cdot b \otimes \beta_x) = r \cdot ab \otimes \beta_x\,, \qquad x \in M, \quad r \in \pi^{-1}(x), \quad a, b \in F$$

This bilinear morphism induces a bilinear map (multiplication)

$$\Gamma(E_F)\times\Gamma(E_F\otimes T^*M)\to\Gamma(E_F\otimes T^*M),\qquad (v,f)\rightsquigarrow v\cdot f$$

by pointwise multiplication. Again from Palais (1968), Corollary 9.7, one gets the following result:

Theorem 4.4. For $v\in H^k(E_F)$ and $f\in H^J(E_F\otimes T^*M)$, $0\le J\le k$, $vf\in H^J(E_F\otimes T^*M)$; moreover the multiplication map

$$H^k(E_F)\times H^J(E_F\otimes T^*M)\to H^J(E_F\otimes T^*M),\qquad (v,f)\rightsquigarrow vf$$

is continuous.

Remark. By the preceding theorem there is a well-defined smooth map

$$\mathscr{G}^{k+1}\times H^J(E_F\otimes T^*M)\to H^J(E_F\otimes T^*M),\qquad (g,f)\rightsquigarrow gfg^{-1}$$

and, as in the remark after Corollary 4.3, the canonical scalar product $(\,|\,)_0$ on $H^0(E_F\otimes T^*M)$ is $\mathscr{G}^{k+1}$-invariant, while on each $H^J(E_F\otimes T^*M)$, $0\le J\le k$, we can choose a $\mathscr{G}^{k+1}$-invariant scalar product.

Now consider equation (3); $\nabla_{\mathcal{O}}\colon C^\infty(E_F)\to C^\infty(E_F\otimes T^*M)$ is a differential operator of order 1, then its extends by continuity to a continuous linear map $H^{k+1}(E_F)\to H^k(E_F\otimes T^*M)$ (Palais, 1965, 1968). This extension will be denoted again by $\nabla_{\mathcal{O}}$. Moreover, the multiplication map of Theorem 4.4 is the unique continuous extension of the corresponding multiplication for spaces of C^∞ sections, Hence we conclude that $\mathcal{N}_{\mathcal{O}}$ extends uniquely to a continuous action $\mathscr{D}^k\times\mathscr{G}^{k+1}\to\mathscr{D}^k$, which we shall denote again by $\mathcal{N}_{\mathcal{O}}$, so

$$\mathcal{N}_{\mathcal{O}}(B,g)=:B\cdot g=g^{-1}Bg+g^{-1}\nabla_{\mathcal{O}}g,\qquad g\in\mathscr{G}^{k+1},\quad B\in\mathscr{D}^k \tag{4}$$

Then we can define a continuous action of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$, denoted again $\mathcal{M}$, by requiring that the diagram

$$\begin{array}{ccc}
\mathscr{C}^k\times\mathscr{G}^{k+1} & \xrightarrow{\ \mathcal{M}\ } & \mathscr{C}^k\\
\Big\downarrow{\scriptstyle \varphi_{\mathcal{O}}\times 1} & & \Big\downarrow{\scriptstyle \varphi_{\mathcal{O}}}\\
\mathscr{D}^k\times\mathscr{G}^{k+1} & \xrightarrow{\ \mathcal{N}_{\mathcal{O}}\ } & \mathscr{D}^k
\end{array}$$

commutes. The action $\mathcal{M}$ so defined will be said to be the extended gauge group action; it is the unique continuous extension of $\mathcal{M}\colon\mathscr{C}\times\mathscr{G}\to\mathscr{C}$. From the preceding commutative diagram one obtains for $\mathcal{M}$ the expression

$$\mathcal{M}(A,g)_\sigma=:(A\cdot g)_\sigma=A_\sigma+g_\sigma^{-1}\tilde{\nabla}_A g_\sigma \tag{5}$$

$$A\in\mathscr{C}^k,\qquad g\in\mathscr{G}^{k+1},\qquad g_\sigma=g\,|\operatorname{dom}\sigma,\qquad \sigma\in\Sigma$$

where

$$\tilde{\nabla}_A\colon\quad H^{k+1}(E_F)\to H^k(E_F\otimes T^*M)$$

is the continuous linear map given by

$$\tilde{\nabla}_A(w) = \varphi_{\mathcal{O}}(A)w - w\varphi_{\mathcal{O}}(A) + \nabla_{\mathcal{O}}(w), \qquad w \in H^{k+1}(E_F)$$

This map is in fact independent of $\mathcal{O}$ and for $A \in \mathscr{C}$ one has $\tilde{\nabla}_A = \nabla_A$.

Since $k > \frac{1}{2}\dim M + 1$, from the Sobolev embedding theorem [in an intrinsic formulation, see Palais (1968)] it follows that there are the inclusions

$$\mathscr{G}^{k+1} \subset C^2(E_F), \qquad \mathscr{D}^k \subset C^1(E_g \otimes T^*M)$$

and the corresponding inclusion maps are continuous. Hence, by definition of $\mathscr{C}^k$ (compare Section 4.2), if $A \in \mathscr{C}^k$, then $A_\sigma \in C^1(E_g|\text{dom}\,\sigma \otimes T^*(\text{dom}\,\sigma))$. Moreover, again by definition of $\mathscr{C}^k$, one sees that the A_σ satisfy analog of condition (c) of Section 4.2. From this one sees that if α' is the inverse of the map $\alpha: \mathscr{C}_2 \to \mathscr{C}$ of Section 3, then α' can be extended to a map $\alpha'^k: \mathscr{C}^k \to \mathscr{C}_2^k$, where $\mathscr{C}_2^k$ consists of g-valued 1-forms on R of class C^1, which satisfy the same conditions defining the (smooth) connection forms of P. Then for any $A \in \mathscr{C}^k$, $\alpha'^k(A)$ defines as in the smooth case a covariant differential $\nabla_A: C^\infty(E_F) \to C^1(E_F \otimes T^*M)$. The ∇_A is a generalized differential operator in the sense of Cantor (1981), and, under our assumption on k, it can be extended to a continuous linear map $\nabla_A: H^{k+1}(E_F) \to H^k(E_F \otimes T^*M)$. Thus, recalling the map $\tilde{\nabla}_A$ introduced in (5), we have $\tilde{\nabla}_A = \nabla_A$ for all $A \in \mathscr{C}^k$.

Further, given $\omega \in \mathscr{C}_2^k$ and $r \in R$, one can define the holonomy group $\phi(r)$ of ω as in the C^∞ case and the theory of holonomy groups of a smooth connection form can be applied to holonomy groups of $\omega \in \mathscr{C}_2$. Thus, the results of Section 3 concerning the relations between isotropy groups and holonomy groups can be extended to the action $\mathscr{C}^k \times \mathscr{G}^{k+1} \to \mathscr{C}^k$ here defined.

In particular, consider for any $r \in R$ the homomorphism $\chi_r: \mathscr{G}^{k+1} \to G$ such that $r \cdot \chi_r(g) = g \circ p(r)$; if I_A is the isotropy subgroup of $\mathscr{G}^{k+1}$ at $A \in \mathscr{C}^k$ and $\phi(r)$ is the holonomy group of $\omega = \alpha'^k(A)$, then χ_r maps isomorphically I_A onto the centralizer $\mathscr{Z}(\phi(r))$ of $\phi(r)$ in G. Here I_A is a subgroup of a Lie group; it is clear that it is a compact subgroup of $\mathscr{G}^{k+1}$. Then it is a Lie subgroup of $\mathscr{G}^{k+1}$ and χ_r: $I_A \to \mathscr{Z}(\phi(r))$ is an isomorphism of Lie groups. Notice that the center $\mathscr{Z}(G)$ of G is always a (normal) Lie subgroup of $\mathscr{Z}(\phi(r))$. Then I_A always contains as normal Lie subgroup $\chi_r^{-1}(\mathscr{Z}(G))$. One sees easily that $g \in \chi_r^{-1}(\mathscr{Z}(G))$ iff $g(r) \in \mathscr{Z}(r \cdot G) = r \cdot \mathscr{Z}(G)$ for every $r \in R$. Hence $\chi_r^{-1}(\mathscr{Z}(G)) = \mathscr{Z}(\mathscr{G}^{k+1})$. It follows that, unless G has trivial center, the action of $\mathscr{G}^{k+1}$ in $\mathscr{C}^k$ is never free [because I_A contains $\mathscr{Z}(\mathscr{G}^{k+1})$].

A connection $A \in \mathscr{C}^k$ is said to be generic if $I_A = \mathscr{Z}(\mathscr{G}^{k+1}) \approx \mathscr{Z}(G)$. In general, the set $\mathscr{C}_0^k$ of generic connections is a proper subset of $\mathscr{C}^k$. Still, it can be proven that $\mathscr{C}_0^k$ is an open dense subset of $\mathscr{C}^k$ (see Kondracki and Rogulski, 1983).

5. ORBIT MANIFOLDS INDUCED BY THE EXTENDED GAUGE GROUP ACTION

One cannot expect, in general, that there exists the orbit manifold for the action of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$. Thus, to define a gauge-free configuration manifold for Yang–Mills theories, one way is to restrict the action of $\mathscr{G}^{k+1}$ to the submanifold $\mathscr{C}_0^k \subset \mathscr{C}^k$ of generic connections; if $\mathscr{Z}$ is the center of $\mathscr{G}^{k+1}$ then the induced action of $\mathscr{G}^{k+1}/\mathscr{Z}$ on $\mathscr{C}_0^k$ is then free and the orbit manifold $\mathscr{C}_0^k/(\mathscr{G}^{k+1}/\mathscr{Z})$ does exist (Narasimhan and Ramadas, 1979; Mitter and Viallet, 1981). Another way, more suitable for physical applications, is to introduce the subgroup $\mathscr{G}_*^{k+1} \subset \mathscr{G}^{k+1}$ of pointed gauge transformations (i.e., fixing a base point $*$ of $\mathscr{C}^k$). Now $\mathscr{G}_*^{k+1}$ acts freely on the whole manifold $\mathscr{C}^k$ and again it can be shown that the orbit manifold $\mathscr{C}^k/\mathscr{G}_*^{k+1}$ exists (Narasimhan and Ramadas, 1979; Mitter and Viallet, 1981). A more satisfactory approach seems to be the one introduced by Kondracki and Rogulski (1983). Here the global structure of the topological space $\mathscr{C}^k/\mathscr{G}^{k+1}$ is considered and it is shown that this space admits a stratification into Hibertable manifolds that are the orbit manifolds for the action of $\mathscr{G}^{k+1}$ on the members of a countable family of $\mathscr{G}^{k+1}$-invariant submanifolds of $\mathscr{C}^k$, including the open submanifold $\mathscr{C}_0^k$ of generic connections. This result can be improved; in fact, the above orbit manifolds carry a natural weak Riemannian structure, as will be shown in the next section.

In this section we report the main steps for the construction of these orbit manifolds in the framework introduced in the preceding sections. We have

$$\mathscr{Z} = \{g \in \mathscr{G}: \quad \forall x \in M, \quad g(x) = s, \quad s \in \mathscr{Z}(G)\} \tag{6}$$

by identifying $\mathscr{Z}(G)$ with a subgroup of $(E_G)_x$, $\forall x \in M$.

We shall need the following decomposition theorem:

Theorem 5.1. For $A \in \mathscr{C}^k$ one has

$$\mathscr{D}^k = \nabla_A(H^{k+1}(E_g)) \oplus \operatorname{Ker} \nabla_A^*$$

where the closed subspaces $\nabla_A(H^{k+1}(E_g))$ and $\operatorname{Ker} \nabla_A^*$ of $\mathscr{D}^k$ are orthogonal for the H^0-scalar product. [The notion of formal adjoint is given here for a generalized differential operator in the sense of Cantor (1981).]

Remark. For $A \in \mathscr{C}$ the decomposition of the theorem is a standard result of the theory of elliptic operators, since in this case ∇_A is a differential operator with injective symbol. For $A \in \mathscr{C}^k - \mathscr{C}$ the decomposition follows directly from the generalization of these results to (elliptic) differential operators with continuous coefficients [Cantor (1981), Theorem 3.13]. The meaning of this decomposition theorem becomes clear by considering the identification map

$$(\varphi_A^{-1} \times 1_{\mathscr{D}^k}) \cdot T\varphi_A\colon \quad T\mathscr{C}^k \to \mathscr{C}^k \times \mathscr{D}^k \tag{7}$$

(which does not depend on $A \in \mathscr{C}^k$ chosen).

With this identification, if we pose

$$f_A\colon \quad \mathscr{G}^{k+1} \to \mathscr{C}^k, \qquad g \rightsquigarrow A \cdot g$$

we have

$$T_e f_A\colon \quad H^{k+1}(E_g) \to \mathscr{D}^k, \qquad T_e f_A = \nabla_A$$

[for the identification $T_e \mathscr{G}^{k+1} \approx H^{k+1}(E_g)$ see Section 4.1]. Then Theorem 5.1 tells us that the image of $T_e f_A$ in $\mathscr{D}^k$ admits a topological supplement (and the same holds for $T_g f_A$, $g \in \mathscr{G}^{k+1}$). Incidentally, we remark that, thanks to (7), to give $\mathscr{C}^k$ a Riemannian structure it is sufficient to define a scalar product on $\mathscr{D}^k$: in Section 4.2 we have seen that there exists a canonical $\mathscr{G}^{k+1}$-invariant scalar product on $H^0(E_g \otimes T^*M)$ and that we can also choose a $\mathscr{G}^{k+1}$-invariant scalar product on $H^k(E_g \otimes T^*M)$: so we shall regard $\mathscr{C}^k$ as a weak or strong Riemannian manifold for the metrics associated with these products.

Now we quote two results [see, for instance, Kondracki and Rogulski (1983) Theorems 2.4.9 and 3.2.1) on the action of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$ technically relevant in connection with the slice theorem below.

Theorem 5.2. The action of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$ is proper.

One can give a direct proof of this result based on ordinary differential equations methods.

Theorem 5.3. $\forall A \in \mathscr{C}^k$ the orbit $A \cdot \mathscr{G}^{k+1}$ is a submanifold of $\mathscr{C}^k$.

To be more precise, it is possible to see that $\forall A \in \mathscr{C}^k$ the map

$$i_A\colon \quad \mathscr{G}^{k+1}/I_A \to \mathscr{C}^k, \qquad \{\vartheta\} \rightsquigarrow A \cdot \vartheta$$

is a diffeomorphism onto $A \cdot \mathscr{G}^{k+1}$.

By using Theorems 5.2 and 5.3 we have the following "slice theorem" [see Kondracki and Rogulski (1983), Theorem 3.3.4]:

Theorem 5.4. $\forall A \in \mathscr{C}^k$ there exists a tubular neighborhood of $A \cdot \mathscr{G}^{k+1}$ for the action of $\mathscr{G}^{k+1}$ on C^k.

We give an explicit expression for this tubular neighborhood. Let

$$N_A = \{(A', X') \in (T\mathscr{C}^k | (A \cdot \mathscr{G}^{k+1})) \approx A \cdot \mathscr{G}^{k+1} \times \mathscr{D}^k\colon$$
$$(X' | Y)_0 = 0, \quad \forall Y \in T_A (A \cdot \mathscr{G}^{k+1}) \approx V_A \subset \mathscr{D}^k\}$$
$$N_A^\varepsilon = \{(A', X') \in N_A\colon \quad (X' | X')_k < \varepsilon^2\}, \qquad \varepsilon > 0$$

Then $(\lambda, N_A^\varepsilon, \exp|N_A^\varepsilon)$, where λ is the vector bundle $\lambda = (N_A, \tilde{\pi}, A \cdot \mathscr{G}^{k+1})$ and exp is the exponential map associated with the strong metric on $\mathscr{C}^k$, is a tubular neighborhood for the action of $\mathscr{G}^{k+1}$ on $\mathscr{C}^k$.

Now let

$$\mathscr{C}^k_{(S)} = \{A \in \mathscr{C}^k: \quad I_A \in (S)\}$$

where S is any compact Lie subgroup of $\mathscr{G}^{k+1}$ such that there exists at least one gauge potential $A \in \mathscr{C}^k$ with isotropy group S, and (S) is the conjugacy class of S in $\mathscr{G}^{k+1}$. Then $\mathscr{C}^k_{(S)}$ is a submanifold of $\mathscr{C}^k$, clearly $\mathscr{G}^{k+1}$-invariant. A slice theorem holds as well for each submanifold $\mathscr{C}^k_{(S)}$. In this case a tubular neighborhood of $A \in \mathscr{C}^k_{(S)}$ is given by $(\nu, N^\varepsilon_{A(S)}, \exp|N^\varepsilon_{A(S)})$, with $\nu = (N_{A(S)}, \tilde{\pi}, A \cdot \mathscr{G}^{k+1})$ and

$$N_{A(S)} = \{(A', X') \in N_A \,|\, I_{(A',X')} \in (S)\}$$

with $I_{(A',X')}$ the isotropy group of (A', X') for the canonically induced action of $\mathscr{G}^{k+1}$ on $T\mathscr{C}^k$ ($N^\varepsilon_{A(S)}$ as above).

Moreover, let H_A be defined by $N_{A(S)}|\{A\} = \{A\} \times H_A$; then there is a smooth map

$$\chi: \quad N_{A(S)} \to A \cdot \mathscr{G}^{k+1} \times H_A, \qquad (A', X') \rightsquigarrow (A', X)$$

where X is the only element such that $g^{-1}Xg = X'$ for any $g \in \mathscr{G}^{k+1}$ s.t. $A \cdot g = A'$. Next let $\pi: \mathscr{C}^k_{(S)} \to \mathscr{C}^k_{(S)}/\mathscr{G}^{k+1}$ be the canonical projection and let

$$U_A = \exp(N^\varepsilon_{A(S)})$$

Then we have a smooth map

$$\psi_A: \quad \pi(\exp(N^\varepsilon_{A(S)})) =: \pi(U_A) \to H_A$$

$$A' \cdot \mathscr{G}^{k+1} \rightsquigarrow (pr_2 \circ \chi \circ (\exp|N^\varepsilon_{A(S)})^{-1})$$

From the properties of χ one concludes (as in the finite-dimensional case) the following result:

Theorem 5.5. The orbit manifold $\mathscr{R}^k_{(S)} = \mathscr{C}^k_{(S)}/\mathscr{G}^{k+1}$ exists. Furthermore, $(\pi(U_A), \psi_A, H_{A(S)})_{A \in \mathscr{C}^k_{(S)}}$ is an atlas of $\mathscr{R}^k_{(S)}$.

6. WEAK RIEMANNIAN STRUCTURE ON THE ORBIT MANIFOLDS $\mathscr{R}^k_{(S)}$

In this section we define for the case $G = SU(n)$ on each $\mathscr{R}^k_{(S)}$ a weak metric that is naturally related to the "kinematic" part of the Lagragian considered in the heuristic formulation of Yang–Mills theories (see, e.g., Babelon and Viallet, 1981).

Let $\pi : \mathscr{C}^k_{(S)} \to \mathscr{R}^k_{(S)}$ be the projection; since π is a submersion, we can introduce the vector subbundle Ker $T\pi$ of $T\mathscr{C}^k_{(S)}$ (see Bourbaki, 1971, 7.5.5). Denoting by the same symbol the total space of Ker $T\pi$, we have

$$\text{Ker } T\pi = \bigcup_{A \in \mathscr{C}^k_{(S)}} \{A\} \times T_A(A \cdot \mathscr{G}^{k+1})$$

The usual proof of the existence of the normal bundle for strong Riemannian manifolds fails in general in the case of weak ones. So we shall give in some detail the proof of the following:

Theorem 6.1. The triple $\nu = (\text{Ker } T\pi^{\perp_0}, \pi | \text{Ker } T\pi^{\perp_0}, \mathscr{C}^k_{(S)})$,[2] where

$$\text{Ker } T\pi^{\perp_0} = \bigcup_{A \in \mathscr{C}^k_{(S)}} \{A\} \times (T_A(A \cdot G^{k+1})^{\perp_0})$$

is a vector subbundle of $T\mathscr{C}^k_{(S)}$. Briefly, Ker $T\pi$ has a weak normal bundle.

Proof. Let $\mathscr{I} : T\mathscr{C}^k \to \mathscr{C}^k \times \mathscr{D}^k$, $(A, X) \rightsquigarrow (A, \mathscr{I}_A X)$ be the canonical identification and

$$W_{(S)} = \bigcup_{A \in \mathscr{C}^k_{(S)}} \{A\} \times \mathscr{I}_A(T_A \mathscr{C}^k_{(S)}) =: \bigcup_{A \in \mathscr{C}^k_{(S)}} W_{(S)A}$$

$$\mathscr{V}_{(S)} = \bigcup_{A \in \mathscr{C}^k_{(S)}} \{A\} \times \mathscr{I}_A(T_A(A \cdot \mathscr{G}^{k+1})) =: \bigcup_{A \in \mathscr{C}^k_{(S)}} \{A\} \times V_A$$

Since $T\mathscr{C}^k_{(S)}$ is a submanifold of $T\mathscr{C}^k$ and Ker $T\pi$ is a submanifold of $T\mathscr{C}^k_{(S)}$, one can give $W_{(S)}$ and $\mathscr{V}_{(S)}$ a differentiable structure such that (i) $\mathscr{I}$ maps diffeomorphically $T\mathscr{C}^k_{(S)}$ on $W_{(S)}$ and Ker $T\pi$ on $\mathscr{V}_{(S)}$, and (ii) $\mathscr{V}_{(S)}$ is a submanifold of $W_{(S)}$, which in turn is a submanifold of $\mathscr{C}^k \times \mathscr{D}^k$.

Moreover,

$$\mathscr{I}_A(T_A(A \cdot \mathscr{G}^{k+1})^{\perp_0}) = [\mathscr{I}_A(T_A(A \cdot \mathscr{G}^{k+1})]^{\perp_0}, \qquad A \in \mathscr{C}^k$$

But to prove that ν is a subbundle is equalent to showing that

$$\mathscr{P}_{(S)}: \quad T\mathscr{C}^k_{(S)} \to \text{Ker } T\pi, \qquad (A, X) \rightsquigarrow (A, \mathscr{P}_{(S)A}(X))$$

[where $\mathscr{P}_{(S)A}$: $T_A\mathscr{C}^k_{(S)} \to T_A(A \cdot \mathscr{G}^{k+1})$ is the weak orthogonal projection with kernel $T_A(A \cdot \mathscr{G}^{k+1})^{\perp_0}$] is a morphism of vector bundles. This, in turn, is true iff $P_{(S)} = \mathscr{I} \circ \mathscr{P}_{(S)} \circ (\mathscr{I} | T\mathscr{C}^k_{(S)})^{-1}$ is a morphism of vector bundles. Now consider the inclusion maps $\mathscr{E} : W_{(S)} \to \mathscr{C}^k_{(S)} \times \mathscr{D}^k$ and $\mathscr{E}' : \mathscr{V}_{(S)} \to \mathscr{C}^k_{(S)} \times \mathscr{D}^k$, which are smooth because $W_{(S)}$ and $\mathscr{V}_{(S)}$ are submanifolds. It is easy to check that one has

$$P \circ \mathscr{E} = \mathscr{E}' \circ P_{(S)}$$

where $P : \mathscr{C}^k_{(S)} \times \mathscr{D}^k \to \mathscr{C}^k_{(S)} \times \mathscr{D}^k$ is defined by $P(A, X) = (A, P_A(X))$, P_A being the weak orthogonal projection with range V_A in $\mathscr{D}^k$. One concludes that $P_{(S)}$ is smooth iff P is smooth.

[2] $\perp_0$ denotes weak orthocomplementation.

Lemma 6.2. The map

$$\mathscr{C}^k_{(S)} \to L(\mathscr{D}^k, \mathscr{D}^k), \qquad A \rightsquigarrow P_A$$

is smooth [uniform topology on $L(\mathscr{D}^k, \mathscr{D}^k)$].

Proof. The proof is in several steps. (a) First one observes that $\forall A \in \mathscr{C}^k_{(S)}$, $P_A : \mathscr{D}^k \to \mathscr{D}^k$ may be written in the form

$$P_A = \nabla_A G_A \nabla^*_A \tag{8}$$

where G_A is a kind of Green operator for the "Laplacian" $\Delta_A = \nabla^*_A \nabla_A$: $H^{k+1}(E_g) \to H^{k+1}(E_g)$. In fact, since Δ_A is elliptic self-adjoint, one has the following decomposition:

$$\begin{aligned} H^{k+1}(E_g) &= \Delta_A(H^{k+1}(E_g)) \oplus \operatorname{Ker} \Delta_A | H^{k-1}(E_g) \\ &=: \mathscr{V}^{k-1}_A \oplus \mathscr{H}^{k-1}_A \end{aligned}$$

generalizing the classical one (Cantor, 1981, Theorem 3.13). Hence, if we pose $\mathscr{H}^{k+1}_A = \operatorname{Ker} \Delta_A | H^{k+1}(E_g)$ and $\Delta'_A = \Delta_A | (\mathscr{H}^{k+1}_A)^\perp$ [the orthogonal complement of $\mathscr{H}^{k+1}_A$ is taken in $H^{k+1}(E_g)$], then Δ'_A is an isomorphism onto $\mathscr{V}^{k-1}_A$ (by open mapping theorem). Next, define

$$\begin{aligned} G_A\colon \quad & \mathscr{V}^{k-1}_A \to (\mathscr{H}^{k+1}_A)^\perp \qquad \text{(as above)} \\ & G_A = (\Delta'_A)^{-1} \end{aligned}$$

Then (8) is readily verified by checking that P_A and $\nabla_A G_A \nabla^*_A$ agree on the subspaces $\operatorname{Ker} \nabla^*_A | \mathscr{D}^k$ and $\nabla_A(H^{k+1}(E_g))$ of $\mathscr{D}^k$ and so they agree on the whole $\mathscr{D}^k$ (Theorem 5.1).

(b) The following construction is useful. Let $\bar{A} \in \mathscr{C}^k_{(S)}$, $\bar{A} \cdot \mathscr{G}^{k+1}$ its orbit in $\mathscr{C}^k_{(S)}$. Since $i_{\bar{A}}: \mathscr{G}^{k+1}/I_{\bar{A}} \to \bar{A} \cdot \mathscr{G}^{k+1}$ is a diffeomorphism (Theorem 5.3) and $(\mathscr{G}^{k+1}, \pi', \mathscr{G}^{k+1}/I_{\bar{A}}, I_{\bar{A}})$ is a principal bundle (Bourbaki, 1971, Theorem 6.2.4), there are an open neighborhood T of $\bar{A}$ in $\bar{A} \cdot \mathscr{G}^{k+1}$ and a smooth map $\xi: T \to \mathscr{G}^{k+1}$ such that

$$\bar{A} \cdot \xi(A') = A', \qquad \forall A' \in \bar{A} \cdot \mathscr{G}^{k+1} \tag{9}$$

We shall pose

$$\begin{aligned} \lambda\colon \quad & \exp(N^\varepsilon_{\bar{A}(S)} | T) = \mathscr{O} \to \mathscr{G}^{k+1} \\ & \tilde{A} \rightsquigarrow (\xi \circ \tilde{\pi} \circ (\exp | N^\varepsilon_{A(S)})^{-1})(\tilde{A}) \\ & (\tilde{\pi}\colon \quad N^\varepsilon_{\bar{A}(S)} \to \bar{A} \cdot \mathscr{G}^{k+1}) \end{aligned}$$

(c) With some calculations one verifies the following formulas:

$$\operatorname{Ad}_{k\pm1}(\vartheta)(\mathscr{H}^{k\pm1}_{A'}) = \mathscr{H}^{k\pm1}_{A'\cdot\vartheta^{-1}}, \qquad \forall \vartheta \in \mathscr{G}^{k+1}, \quad \forall A' \in \mathscr{C}^k \tag{10}$$

where

$$\begin{aligned} \mathrm{Ad}_{k\pm1}(\vartheta)\colon \quad & H^{k\pm1}(E_g) \to H^{k\pm1}(E_g) \\ & X \rightsquigarrow \vartheta X \vartheta^{-1}, \qquad \forall \vartheta \in \mathcal{G}^{k+1} \\ & \mathcal{H}_{A'}^{k+1} = \mathrm{Lie}(I_{A'}), \qquad \forall A' \in \mathcal{C}^k \end{aligned} \tag{11}$$

In particular, $\forall A' \in \mathcal{O}$,

$$\mathrm{Ad}_{k\pm1}(\lambda(A'))(\mathcal{H}_{A'}^{k\pm1}) \overset{(10)}{=} \mathcal{H}_{A'\cdot\lambda(A')^{-1}}^{k\pm1} \overset{(11)}{=} \mathrm{Lie}(I_{A'\cdot\lambda(A')^{-1}}) \tag{12}$$

Finally one checks, by using tubular neighborhood properties, that

$$I_{A'\cdot\lambda(A')^{-1}} = I_{\bar{A}} \tag{13}$$

So, from (12) and (13), it follows that

$$\mathrm{Ad}_{k\pm1}(\lambda(A'))(\mathcal{H}_{A'}^{k\pm1}) = \mathcal{H}_{\bar{A}}^{k\pm1}, \qquad \forall A' \in \mathcal{O}$$

and, by $\mathcal{G}^{k+1}$-invariance of weak and strong inner products on $H^{k\pm1}(E_g)$, we can conclude, $\forall A' \in \mathcal{O}$,

$$\begin{aligned} \mathrm{Ad}_{k-1}(\lambda(A'))(\mathcal{V}_{A'}^{k-1}) &= \mathcal{V}_{\bar{A}}^{k-1} \\ \mathrm{Ad}_{k+1}(\lambda(A'))[(\mathcal{H}_{A'}^{k+1})^{\perp}] &= (\mathcal{H}_{\bar{A}}^{k+1})^{\perp} \end{aligned} \tag{14}$$

(d) By (8) part (c) we can write

$$\begin{aligned} P_{A'} = \nabla_{A'} \circ \mathrm{Ad}_{k+1}(\lambda(A')^{-1} \circ P_{(\mathcal{H}_{\bar{A}}^{k-1})^{\perp}} \circ \mathrm{Ad}_{k+1}(\lambda(A')) \circ G_{A'} \\ \circ P_{\mathcal{V}_{A'}^{k-1}} \circ \mathrm{Ad}_{k-1}(\lambda(A')^{-1}) \circ P_{\mathcal{V}_{\bar{A}}^{k-1}} \circ \mathrm{Ad}_{k-1}(\lambda(A')) \circ \nabla_{A'}^{*} \end{aligned} \tag{15}$$

where

$$P_{\mathcal{V}_{\bar{A}}^{k-1}}\colon H^{k-1}(E_g) \to \mathcal{V}_{\bar{A}}^{k-1}$$

is the orthogonal projection onto $\mathcal{V}_{\bar{A}}^{k-1}$, [and analogously for $P_{\mathcal{V}_{A'}^{k-1}}$ and $P_{(\mathcal{H}_{\bar{A}}^{k+1})^{\perp}}$]. It is now easy to see that $\nu\colon \mathcal{O} \to L(\mathcal{D}^k, \mathcal{D}^k)$, $A' \rightsquigarrow P_{A'}$, is smooth. In fact, the smoothness of

$$\begin{aligned} &: \mathcal{O} \to L(\mathcal{D}^k, H^{k-1}(E_g)), \qquad A' \rightsquigarrow \nabla_{A'}^{*} \\ &: \mathcal{O} \to L(H^{k+1}(E_g), \mathcal{D}^k), \qquad A' \rightsquigarrow \nabla_{A'} \end{aligned}$$

follows directly from the explicit expressions of $\nabla_{A'}$ and $\nabla_{A'}^{*}$; on the other hand, for the map

$$\begin{aligned} \nu\colon \quad & \mathcal{O} \to L(\mathcal{V}_{\bar{A}}^{k-1}, (\mathcal{H}_{\bar{A}}^{k+1})^{\perp}) \\ & A' \rightsquigarrow P_{(\mathcal{H}_{\bar{A}}^{k+1})^{\perp}} \circ \mathrm{Ad}_{k+1}(\lambda(A')) \circ G_{A'} \\ & \qquad \circ P_{\mathcal{V}_{A}^{k-1}} \circ \mathrm{Ad}_{k-1}(\lambda(A')^{-1}) \circ i_{\mathcal{V}_{A}^{k-1}} \end{aligned}$$

where

$$i_{\mathcal{V}_{\bar{A}}^{k-1}}\colon \mathcal{V}_{\bar{A}}^{k-1} \to H^{k-1}(E_g)$$

is the inclusion map, one can observe that $\nu = \mathrm{Inv} \circ \nu'$, where

$$\nu' : \mathcal{O} \to L(\mathcal{H}_{\bar{A}}^{k+1})^{\perp}, \mathcal{H}_{\bar{A}}^{k-1})$$

$$A' \rightsquigarrow P_{\mathcal{V}_{\bar{A}}^{k-1}} \circ \mathrm{Ad}_{k-1}(\lambda(A')) \circ \Delta_{A'} \circ \mathrm{Ad}_{k+1}(\lambda(A')^{-1} \circ i_{(\mathcal{H}_{\bar{A}}^{k+1})^{\perp}}$$

and

$$i_{(\mathcal{H}_{\bar{A}}^{k+1})^{\perp}}: \quad (\mathcal{H}_{\bar{A}}^{k+1})^{\perp} \to H^{k+1}(E_g)$$

is the inclusion map, and Inv is the smooth map $T \rightsquigarrow T^{-1}$ defined on the open set of the invertible operators in $L((\mathcal{H}_{\bar{A}}^{k+1})^{\perp}, \mathcal{V}_{\bar{A}}^{k-1})$.

This completes the proof of Theorem 6.1 and now we are ready to construct a weak metric on $\mathcal{R}_{(S)}^{k}$. Namely, if there exists a map

$$\tilde{g}: \quad T\mathcal{R}_{(S)}^{k} \times_{\mathcal{R}_{(S)}^{k}} T\mathcal{R}_{(S)}^{k} \to \mathcal{R}_{(S)}^{k} \times \mathbb{R}$$

such that

$$\tilde{g} \circ (T\pi \times_{\mathcal{C}_{(S)}^{k}} T\pi) = (\pi \times 1_{\mathbb{R}}) \circ g \,|\, \nu \times_{\mathcal{C}_{(S)}^{k}} \nu \tag{16}$$

where

$$g: \quad T\mathcal{C}_{(S)}^{k} \times_{\mathcal{C}_{(S)}^{k}} T\mathcal{C}_{(S)}^{k} \to \mathcal{C}_{(S)}^{k} \times \mathbb{R}$$

is the weak metric induced on $\mathcal{C}_{(S)}^{k}$ from $\mathcal{C}^{k}$, then this map is unique and smooth, fiber-preserving, and bilinear on each fiber. Such a map does exist, as one verifies by setting, for a fixed $a \in \mathcal{R}_{(S)}^{k}$ and any $A \in \pi^{-1}(a)$

$$g(T\pi(A, X), T\pi(A, Y)) = (\pi \times 1_{\mathbb{R}})\tilde{g}((A, X), (A, Y)),$$
$$\forall (A, X), (A, Y) \in \nu \,|\, A$$

and by checking that it is a well-posed definition because it does not depend on $A \in \pi^{-1}(a)$. We have thus proven the following result:

Corollary 6.3. There exists one and only one metric $\tilde{g}$ on $\mathcal{R}_{(S)}^{k}$ satisfying (16). We shall consider $\mathcal{R}_{(S)}^{k}$ as a (weak) Riemannian manifold for $\tilde{g}$.

7. SOME REMARKS ON THE GEODESIC FLOW

7.1. Geodesic Spray

The weak metric set up in Section 6 is naturally related to the "kinetic" part of the Lagrangian considered in the heuristic formulation of Yang-Mills theories (Babelon and Viallet, 1981) and so it is worthwhile to study the integral flow of its geodesic field. The local expression for $g': \mathcal{R}_{(S)}^{k} \to \mathrm{Met}(T\mathcal{R}_{(S)}^{k})$,

$$g'(a)((a, X)(a, Y)) = (pr_2 \circ g)((a, X), (a, Y))$$

in a chart $(\pi(U_A), \psi_A, H_A)$ for $\mathscr{R}^k_{(S)}$ and the canonically induced chart on $T\mathscr{R}^k_{(S)} \times_{\mathscr{R}^k_{(S)}} T\mathscr{R}^k_{(S)}$ is

$$\begin{aligned} &\mathscr{G}: \quad H^\varepsilon_A \to L^2(H_A) \qquad (=: \text{bilinear forms on } H_A) \\ &\mathscr{G}(X)(Y, Z) = (Y - P_{\exp(A,X)} Y | Z - P_{\exp(A,X)} Z)_0 \end{aligned} \tag{17}$$

(principal part). This equation can be obtained by direct computation, taking account of the equation

$$\mathscr{I}_{A'}(T_{A'}\mathscr{C}^k_{(S)}) = V_{A'} \oplus H_{A'} \qquad \forall A' \in \mathscr{C}^k_{(S)}$$

Equation (17) gives, in a standard way, the equation for the geodesic spray $f: T\mathscr{R}^k_{(S)} \to T(T\mathscr{R}^k_{(S)})$. Locally, if we pose

$$T(T\psi_A) \circ f \circ T\psi_A^{-1} = ((pr_1, F_1)(pr_1, F_2))$$

$$F_1, F_2: \quad H^\varepsilon_A \times H_A \to H_A$$

then F_1 and F_2 must satisfy, $\forall Z_1, Z_2 \in H_A$, the equation

$$\begin{aligned} &D\mathscr{G}|_X(F_1(X, Y))(Y, Z_1) - D\mathscr{G}|_X(Z_1)(Y, F_1(X, Y) \\ &\quad + \mathscr{G}(X)(Z_1, F_2(X, Y)) - \mathscr{G}(X)(F_1(X, Y), Z_2) \\ &= -\tfrac{1}{2} D\mathscr{G}|_X(Z_1)(Y, Y) - \mathscr{G}(X)(Y, Z_2)(X, Y) \in T\psi_A(T[\pi(U_A)]) \\ &= H^\varepsilon_A \times H_A \end{aligned} \tag{18}$$

In (18) the left-hand side is just the local expression of the canonical 2-form on $T\mathscr{R}^k_{(S)}$. Since this is a weak 2-form (Chernoff and Marsden, 1974, p. 9, Theorem 5), it is not granted *a priori* that (18) has solutions $F_1(X, Y)$ and $F_2(X, Y)$ for every $(X, Y) \in H^\varepsilon_A \times H_A$. Attempting to solve directly equation (18) leads to considerable computational difficulties, which can be avoided by observing that at the points $(0, Y) \in \{0\} \times H_A$, F_1 and F_2 are simply given by $F_1(0, Y) = Y$ and $F_2(0, Y) = 0$. One concludes that there is a map $f: T\mathscr{R}^k_{(S)} \to T(T\mathscr{R}^k_{(S)})$ (defined everywhere in $T\mathscr{R}^k_{(S)}$) satisfying pointwise the standard geodesic spray equation; namely

$$\begin{aligned} &T(T\psi_A) \circ f \circ T\psi_A^{-1}(0, Y) = ((0, Y)(Y, 0)) \\ &\forall Y \in H_A, \qquad \forall A \in \mathscr{C}^k_{(S)} \end{aligned} \tag{19}$$

Now we claim that the map f is smooth, thus giving an everywhere defined geodesic spray. Again a direct proof of the smoothness of f is somewhat difficult at this stage, since the transition functions between the charts

$$(T(T\psi_A), T(T(p(U_A)), H^\varepsilon_A \times H_A \times H_A \times H_A)$$

and the whole local expression of f are not simple. The smoothness of f will follow more easily as a by-product of the subsequent developments.

7.2. Geodesic Flow

Now we look for C^∞ paths in $T\mathscr{R}^k_{(S)}$, $u: I \subset \mathbb{R} \to T\mathscr{R}^k_{(S)}$, such that $Tu(E(t)) = f(u(t))$ $\forall t \in I$ (E = canonical vector field on $\mathbb{R}$). Let I be an open interval in $\mathbb{R}$ and let

$$\begin{aligned}
&\tilde{v}: \quad I \to \mathscr{C}^k_{(S)} && \tilde{v}(t) = A + t\tau, \qquad \tau \in \mathscr{D}^k_1\\
&\dot{\tilde{v}}: \quad I \to T\mathscr{C}^k_{(S)} && \dot{\tilde{v}}(t) = T\tilde{v}(E(t))\\
&v: \quad I \to \mathscr{R}^k_{(S)} && v = \pi \circ \tilde{v}\\
&\dot{v}: \quad I \to T\mathscr{R}^k_{(S)} && \dot{v}(t) = Tv(E(t)) = T\pi(\dot{\tilde{v}}(t))
\end{aligned}$$

By some machinery [working pointwise on the fibers of $T\mathscr{R}^k_{(S)}$ and not on $T(T\mathscr{R}^k_{(S)})$ to use $\mathscr{I}_A(T_A\mathscr{C}^k_{(S)}) = V_A \oplus H_A$], one has

$$T\dot{v}(E(t)) = f(\dot{v}(t)) \tag{20}$$

From (20) one concludes that f is smooth. In fact, since

$$\mathscr{I}(\nu) = \bigcup_{A \in \mathscr{C}^k_{(S)}} \{A\} \times H_A \subset \mathscr{C}^k_{(S)} \times \mathscr{D}^k_{(S)}$$

is a submanifold of $\mathscr{C}^k_{(S)} \times \mathscr{D}^k_{(S)}$, the inclusion map $\bar{\mathscr{E}}: \mathscr{I}(\nu) \to \mathscr{C}^k_{(S)} \times \mathscr{D}^k$ induces a C^∞ map

$$T\bar{\mathscr{E}}: T\mathscr{I}(\nu) \to ((\mathscr{C}^k_{(S)} \times \mathscr{D}^k) \times (\mathscr{D}^k \times \mathscr{D}^k))$$

So

$$\begin{aligned}
\tilde{f}: \quad &\mathscr{I}(\nu) \to ((\mathscr{C}^k_{(S)} \times \mathscr{D}^k) \times (\mathscr{D}^k \times \mathscr{D}^k))\\
&(A, X) \rightsquigarrow ((A, X), (X, 0))
\end{aligned}$$

is a smooth vector field on $\mathscr{I}(\nu)$. Now, as it is easy to see, by using (20),

$$T^2\pi \circ (T\mathscr{I}^{-1} \circ T\bar{\mathscr{E}}^{-1} \circ \tilde{f} \circ \mathscr{I}) = f \circ T\pi \,|\, \nu$$

Hence f is smooth and our claim on it is proven. On the other hand, (20) is the starting point for the following result of classification of the geodesics of f:

Proposition 7.1. (i) Let $\tilde{v}: \mathbb{R} \to \mathscr{C}^k$, $\tilde{v}(t) = A + t\bar{\tau}$, $A \in \mathscr{C}^k_{(S)}$, and $\bar{\gamma}(\bar{\tau}) \in H_A$ ($\bar{\gamma}$ is the extension to $\mathscr{D}^k_1$ of $\gamma: \mathscr{D}_1 \to \mathscr{D}$; see Section 4). Let us define $\mathscr{I} = \{t \in \mathbb{R}: \tilde{v}(t) \in \mathscr{C}^k_{(S)}\}$ and I the connected component of $\mathscr{I}$ containing 0. If $I \neq \{0\}$, then $v = \pi \circ \tilde{v} \,|\, I: I \to \mathscr{R}^k_{(S)}$ is a geodesic path in $\mathscr{R}^k_{(S)}$. (ii) Conversely, let $v: J \to \mathscr{R}^k_{(S)}$ be a geodesic path in $\mathscr{R}^k_{(S)}$ Then there exists $\tilde{v}: J \to \mathscr{C}^k_{(S)}$, $\tilde{v}(t) = A + t\bar{\tau}$, such that $v = \pi \circ \tilde{v}$.

7.3. The Generic Stratum

Since the generic stratum $\mathcal{R}_Z^k$ is open and dense in $\mathcal{R}^k$ (Kondracki and Rogulski, 1983, Theorem 4.3.5), it is the natural candidate for the true configuration space of a Lagrangian Yang-Mills theory. So it is of interest to study its geodesic properties. Unfortuantely, a simple analysis of slice properties of $\mathcal{C}^k$ gives:

Theorem 7.2. $\mathcal{R}_Z^k$ is not geodesically complete.

One sees this at once. In fact, for $A \in \mathcal{C}_{(S)}^k$, $S \neq Z$,

$$W = (\exp|N_A^\varepsilon)^{-1}(\mathcal{C}_0^k \cap \exp N_A^\varepsilon) \cap (\{A\} \times \mathcal{D}^k)$$

is open in $\{A\} \times \mathcal{D}^k$. Hence, choosing $(\{A\} \times X) \in W$, $t \to v(t) = \exp(\{A\} \times tX) = A + tX$ is a geodesic path in $\mathcal{C}^k$ such that $v(0) = A \in \mathcal{C}_{(S)}^k$ and $v(t) \in \mathcal{C}_0^k$ for $|t-1|$ sufficiently small. A measure of this noncompleteness is given by the following theorem, which we quote without proof:

Theorem 7.3. Let $\tilde{v}: \mathbb{R} \to \mathcal{C}^k$, $\tilde{v}(t) = A + t\bar{\tau}$, $A \in \mathcal{C}_0^k$, $\bar{\gamma}(\bar{\tau}) \in H_A$, $I \neq \{0\}$ (see Proposition 7.1). Then the set $A = \{t \in \mathbb{R}: \tilde{v}(t) \notin \mathcal{C}_0^k\}$ is nowhere dense in $\mathbb{R}$.

These results seem to indicate that assuming the generic stratum as configuration space could be not justified in general. A better understanding of the meaning of nongeneric strata is presumably required to clarify the rather unpleasant feature that the configuration space cannot in general be described by a simple smooth manifold.

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